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Barriers to Entailment

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Some history

Lloyd Humberstone

Although mostly an accepted platitude, the idea that there is a modal barrier to entailment has sometimes been challenged, especially, as it turns out, in Australia:

> Some people have maintained in print—and many more, I suppose, in casual conversation—that a converse principle also holds: no contingently true statement can entail a necessarily true one, or, more vaguely perhaps, that no contingent statement entails a necessary one. Fortunately, this view, at least in the less vague formulation just given, was decisively disposed of by R. and V. Routley over ten years ago, though it no doubt persists in some quarters. (321)[1]

If you have a look at the Routley-Routley paper, these are the counterexamples they offer:[2]

i) $p \vDash p \vee \neg p$
ii) *Plato exists* $\vDash$ *Plato is Plato*
iii) *Tito knows that* 2+2=4 $\vDash$ 2+2=4.

Val Plumwood (1939–2008)

In addition they raise the issue of the B-axiom in modal logic, which is valid in the standard B and S5 systems: $\phi \rightarrow \Box\Diamond\phi$. This suggests counterexamples like:

iv) $p \vDash \Box\Diamond p$

It would be an embarrassment for the modal barrier thesis if the modal logic S5 provided a genuine counterexample!

Richard Sylvan (1935–1996)

Humberstone concludes that the following barrier thesis is mistaken:

(1) No contingently true statement entails a necessarily true statement.

And yet he still suspects that we haven't reached the heart of the matter. He refines the principle to avoid the $p \vDash p \vee \neg p$ counterexample, getting (*) which (transposed into our own notation) is:

(*) If $\phi \vDash \psi$ and $\nvDash \psi$, then if ϕ is contingent, it is not the case that ψ is necessary.

He writes:

> Now, why have people felt that there is something right about a principle like (*)? I think something along the following lines might be the sort of thing they have in mind, and also that something along these lines might be correct. . . . If ψ is not a valid formula, then it could be false if things turn out in certain ways in various possible worlds. We

[1] Humberstone, I. L. (1982). Necessary conclusions. *Philosophical Studies*, 41:321–335

[2] Routley, R. and Routley, V. (1969). A fallacy of modality. *Noûs*, 3(2):129–153

> need then to know what is going on in *all* possible worlds to be assured that nothing is going to spoil the necessity of the conclusion ψ. So we need premises, all right, but contingent ones, such as the imagined ϕ give us only 'local' information: information about a particular possible world. What we need is 'global' information: information about all worlds. (Humberstone, 1982, 323, notation adapted)

There are some striking similarities between this passage from Humberstone and the informal account of *Particular* and *Universal* we looked at on Day 1.

A Distinction

We need to be more careful distinguishing two things, namely:

E.g. *2+2=4* vs *It is necessary that 2+2=4*.

i) sentences that say things which are necessary (express necessary propositions)
ii) sentences that *say that* something is necessary (express propositions which *attribute* necessity.)

The former expresses a proposition which is necessary, but doesn't attribute necessity to anything.

The latter attributes necessity to the proposition that 2+2=4.

— On Monday we described the modal barrier as saying that you can't get sentences which say how things must be from sentences which merely say how things are.
— That suggests that the conclusion class should be sentences which *attribute* necessity.
— Routley and Routley's first three counterexamples don't directly threaten that barrier
— though the B-based counterexample does
— and counterexamples i) and ii) are easily transformed into arguments that do threaten it directly, namely:

i*) $p \vDash \Box(p \vee \neg p)$
ii*) *Plato exists* $\vDash$ Necessarily, *Plato is Plato*

So it will be interesting to see whether we can establish a modal barrier theorem in the face of these obstacles.

Upshot of this section: we have three Routley-Routley inspired counterexamples to worry about:

$$p \vDash \Box(p \vee \neg p)$$

$$\exists x x = a \vDash \Box(a = a)$$

$$p \vDash \Box\Diamond p$$

Modal Logic

sentence letters:	p, q, r, p_1, p_2 etc.
logical connectives:	$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
modal operators:	$\Box, \Diamond$
punctuation:	(,)

1. If ϕ is a sentence letter, then ϕ is a sentence.
2. If ϕ and ψ are sentences, then
   a. $\neg\phi$ is a sentence.
   b. $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$ are sentences.
   c. $\Box\phi$ and $\Diamond\phi$ are sentences.
3. Nothing else is a sentence.

Models

An ML model is an ordered quadruple,

$$\langle W, R, @, I \rangle$$

where:

1. W is a non-empty set of points (the set of *worlds*)
2. $R \subseteq W^2$ (the *accessibility* relation on worlds)
3. $@ \in W$ (the *actual world*)
4. I is a function mapping each pair of a sentence letter and a world to a member of $\{1, 0\}$ (the interpretation function)

It can help to think of R as a relation of *relative possibility*; $w_1 R w_2$ means that w_2 is possible relative to w_1.

Definition of truth-in-a-model-at-a-world for the modal operators

$V_M(\Diamond\phi, w) = 1$ iff there is $u \in W$ such that wRu and $V_M(\phi, u) = 1$
$V_M(\Box\phi, w) = 1$ iff and only if for all $u \in W$ such that wRu, $V_M(\phi, u) = 1$

Truth in a model and logical consequence

$$V_M(\phi) = 1 \text{ if, and only if, } V_M(\phi, @_M) = 1.$$

A sentence ϕ is an ML-logical consequence of a set of sentences Γ if, and only if, for all M, if $V_M(\Gamma) = 1$, then $V_M(\phi) = 1$.

Truth in a model is truth at the model's actual world. (Just like yesterday, when truth in a TL-model was truth at the model's *present moment*.)

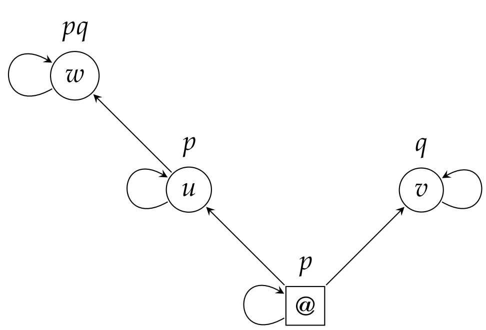

Figure 1: A diagram of a model in which $\Diamond\Diamond p$, $\Diamond p$ and $\Diamond\Diamond(p \wedge q)$ are true, but $\Box p$ is false.

As with $<$ and tense logics, we get different classes of models, and often different modal logics, by placing different restrictions on R.

Logic	R-restrictions	for all w, w_1, w_2...	Characteristic Axiom
K	no restrictions	—	$\Box(p \to q) \to (\Box p \to \Box q)$
D	serial	exists w' such that wRw'	$\Box p \to \Diamond p$
T	reflexive	wRw	$\Box p \to p$
B	reflexive, symmetric	wRw, $w_1 R w_2 \Rightarrow w_2 R w_1$	$p \to \Box\Diamond p$
S4	reflexive, transitive	wRw, if $(wRw_1 \& w_1 R w_2) \Rightarrow w_1 R w_2$	$\Box p \to \Box\Box p$
S5	refl., symm., trans.	all the above	$\Diamond p \to \Box\Diamond p$

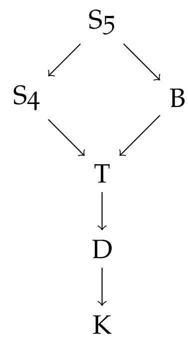

Figure 2: Six modal logics ordered by logical strength, with the strongest at the top.

Two modal barriers

— We can interpret the barrier in two different ways depending on whether we take the particular/universal barrier as a model, or the past/future one.

— Starting with the particular/universal, note that necessity operators are a kind of universal quantifier and consider the relation of *modal model-extension*—adding new worlds and making them accessible from old ones if we choose to.

— The Modally Universal sentences will be the ones that are fragile with respect to modal model-extension. On this approach, we would expect $\Box p$ to be in the barrier's conclusion class. But we wouldn't expect $\Diamond p$ to be in the conclusion class; that would function more like $\exists x Fx$ in the case of the particular/universal barrier.

— A different interpretation of the modal barrier is suggested via analogy with future-switching: think of the actual world as playing the role the past plays in the past/future barrier, and the rest of modal space as playing the role of the future.

— That suggests we consider a kind of "non-actual modal switching" on which the model's interpretation function is permitted to assign different values to sentence letters at non-actual worlds. That would distinguish sentences which are in some way only concerned with goings on at the actual world from sentences whose truth-values depend on what the rest of modal space is up to.

— This gives us two different informal versions of the modal barrier:

Version A You can't get modally universal sentences from sentences which are modally particular.

Version B You can't get claims about how non-actual modal space is from sentences which are only about actual modal space.

— Both are generalisations from which the thesis that you can't get a *must* from an *is* follows, but they offer to subsume that fact under different general patterns: on one, *must*s are modally universal and we can't get modally universal things from particular things. On the other, *must*s are about goings on in non-actual modal worlds, and we can't find out about that from premises which concern the actual modal world alone.

— Either idea can be developed, but today we'll focus on version A (which is the version I think ends up being the closest to the intuitive modal barrier.

The modal barrier—version A

— The idea: modally-universal sentences don't follow form modally-particular ones (*m-universal* and *m-particular* for short.)

— We need to define *modal extension*.

Definition 1 (Modal Extension ($\sqsubseteq$)). *A model N is an m-extension of M if, and only if,*

The intuitive idea is to keep the old possible worlds, but add new ones, making those new worlds accessible from the old or from new ones if we wish (pairs of old worlds stand in the R-relation if and only if they did so in the old model.)

1.

$$W_M \subseteq W_N$$

2. for all $u, v \in W_M$,

$$uR_M v \text{ iff } uR_N v$$

3.

$$@_M = @_N$$

4. for all sentence letters α, and $w \in W_M$,

$$I_M(\alpha, w) = I_N(\alpha, w).$$

Now we use m-extension to define our premise and conclusion classes:

Definition 2 (Modally Particular). *A sentence ϕ is Modally Particular if and only if it is m-extension preserved.*

Definition 3 (Modally Universal). *A sentence, ϕ, is Modally Universal if and only if it is m-extension-fragile.*

Theorem 4 (Modal Barrier Theorem (Version A)). *If Γ is a satisfiable, set of m-particular sentences and ϕ is m-universal, then $\Gamma \not\models \phi$.*

Proof. This is an instance of the General Barrier Theorem we proved yesterday. $\square$

A-version Taxonomy

Modally particular:
$\Diamond p$, $\neg\Box p$, $\Diamond p \wedge \Diamond q$, $\Diamond(p \wedge q)$, $\Diamond(p \vee q)$ and $\Diamond p \vee \Diamond q$, logical truths.
Modally Universal Sentences:
$\Box p$, $\neg\Diamond p$, $p \wedge \Box q$, $p \wedge \Box p$, $\Box(p \wedge q)$, $\Box(p \vee q)$ and $\neg(p \to \Diamond q)$.
Neither (m-particular nor m-universal.)
$p \vee \Box q$.

— The above taxonomical remarks apply independently of restrictions on the accessibility relation, and hence to all six of the logics K, D, T, B, S4, and S5.

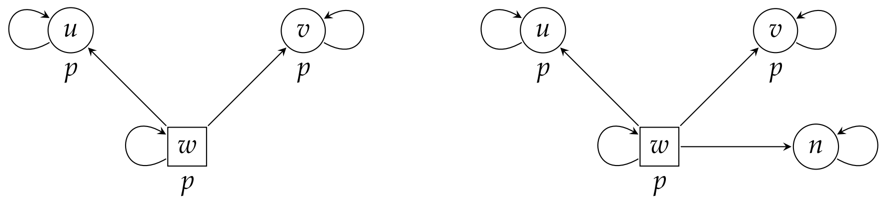

Figure 3: $\Box p$ is m-extension fragile.

- Things get more interesting in this respect when we consider sentences with nested modal operators.
- The status of $\Box\Diamond p$ depends on the logic.
  - In S5 models it is m-extension preserved
  - in S4 models it is m-extension fragile
  - and in B models it is neither.

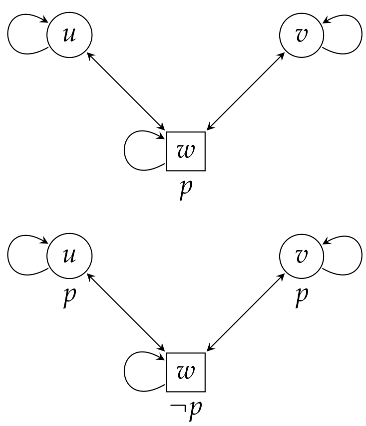

Figure 4: Two ways to make $\Box\Diamond p$ true in a B-model.

Back to the counterexample:

$$p \models \Box\Diamond p$$

For sublogics of S4, the argument is not valid, and so not a counterexample. If the logic is *B*, then $\Box\Diamond p$ is neither m-particular nor m-universal. And if the logic is *S*5, then it is m-particular.

References

Humberstone, I. L. (1982). Necessary conclusions. *Philosophical Studies*, 41:321–335.

Routley, R. and Routley, V. (1969). A fallacy of modality. *Noûs*, 3(2):129–153.