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Barriers to Entailment

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ANU Logic Summer School 2025, Tuesday

Plan for the week:

Monday: Introduction and the Particular/Universal Barrier
Tuesday: Tense Logic and the Past/Future Barrier
Wednesday: Modal Logic and the *is/must* Barrier
Thursday: Complex Logics and the *is/ought* Barrier
Friday: Wrapping up

Other barriers:

— *is/all* barrier
— *was/will* barrier
— *is/must* barrier
— non-indexical/indexical barrier (not controversial)

What happened yesterday:

— the *is/ought* barrier (controversial, counterexamples?)
— But other barriers are susceptible to the counterexamples too
— The Plan: reformulate the barriers in a unified way to avoid the counterexamples, save the *is/ought* barrier, resolve the controversy!
— Approach: refine what we mean by things like *descriptive*, *normative*, *particular*, *universal* etc.
— Started with Particular/Universal barrier
— Underlying idea: Universal sentences are *sensitive* (to be made more precise) to extensions to the model. Particular sentences are not.
— So we define the inter-model relation of *extension*.
— Then define Particular sentences as those which are *preserved* with respect to the relation of extension and Universal sentences as those which are *fragile* with respect to it.
— This gave us a certain taxonomy of the sentences of FOL: e.g. Particular: Fa, Universal: $\forall x Fx$, Neither: $Fa \vee \forall x Gx$ and Both (!): $Fa \wedge \neg Fa$.
— It also allowed us to prove a barrier theorem: If Γ is a satisfiable set of Particular sentences and ϕ is Universal, then $\Gamma \nvDash \phi$.

Some house-keeping from yesterday:

1. What's up with the official definition of **extension**?

   Definition 1 (Model-Extension ($\Subset$))**.** *A model N extends a model M just in case*

   (a) $D_M \subseteq D_N$ *and*
   (b) for all individual constants α,

   $$I_N(\alpha) = I_M(\alpha)$$

   *(c) for all n-place predicates Π **and all** $d_i \in D_M$,*

   $$d_1, ..., d_n \in I_N(\Pi) \text{ iff } d_1, ..., d_n \in I_M(\Pi).$$

2. What about the definition of **fragility**?

   A sentence ϕ of FOL is $\Subset$-fragile iff, for **every** model M, if $V_M(\phi) = 1$, then there is a model N such that $M \Subset N$, $V_N(\phi) = 0$.

   Informal gloss: *whenever* the sentence is true in a model, there is some extension of that model that makes it false.

   Contrast that with $\Subset$-**breakability**:

   A sentence ϕ of FOL is $\Subset$-breakable iff, there is some model M, such that $V_M(\phi) = 1$, and a model N, such that $M \Subset N$, $V_N(\phi) = 0$.

   To see the difference note that

   $$Fa \vee \forall x Gx$$

   is $\Subset$-breakable without being $\Subset$-fragile.

3. One thing that I could have stressed more yesterday: we proved the theorem, so obviously there are no counterexamples. But what *happens* with the putative counterexamples we looked at?

   (a) Prior 1:

   $$Fa \vDash Fa \vee \forall x Gx$$

   (b) Prior 2:

   $$Fa \vee \forall x Gx, \neg Fa \vDash \forall x Gx$$

What's happening today:

— It's Time day!

— We are going to see if we can use the same ideas to formulate and prove the past/future barrier.

> All inferences from experience suppose, as their foundation, that the future will resemble the past . . . if there be any suspicion that the course of nature may change, and that the past may be no rule for the future, all experience becomes useless, and can give rise to no inference or conclusion. It is impossible, therefore, that any argument from experience can prove this resemblance of the past to the future; since all these arguments are founded on the supposition of that resemblance. (Hume, 1748, 4.21/37)

— OK, what do we need:

1. a logic (formal language, set of models, consequence relation) (like FOL in the Particular/Universal case)
2. a binary relation (like extension in the Particular/Universal case)
3. definitions of Past and Future
4. a proof

1. The Logic

Language

sentence letters:	p, q, r, etc.
connectives:	$\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
tense operators:	$\mathcal{F}$, $\mathcal{G}$, $\mathcal{P}$, $\mathcal{H}$

Informally, we can think of
$\mathcal{F}p$ as saying
at some future time p,
$\mathcal{G}p$ as saying
at all future times p,
$\mathcal{P}p$ as saying
at some time in the past p
and $\mathcal{H}p$ as saying
at all past times p.

Sentences

1. If ϕ is a sentence letter, then ϕ is a sentence.
2. If ϕ and ψ are sentences, then
   (a) $\neg\phi$ is a sentence,
   (b) $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are sentences,
   (c) $\mathcal{F}\phi$, $\mathcal{G}\phi$, $\mathcal{P}\phi$, and $\mathcal{H}\phi$ are sentences.
3. Nothing else is a sentence.

Models

A TL model is an ordered quadruple,

$$(T, <, n, I)$$

where:

1. T is a non-empty set (the set of *times*),
2. $< \subseteq T^2$ (the *earlier than* relation),
3. $n \in T$ (the model's 'now' or present)
4. I is an interpretation function mapping pairs of sentence letters and times to truth values (1 represents *true*, 0 *false*.) e.g. $I(q, t_3) = 1$ tells us that q is true at t_3.

We can model assumptions about the structure of time by imposing different restrictions on the $<$-relation.

We'll use a strong logic to illustrate and assume that $<$ is transitive, anti-symmetric, R-total, L-total, R-extendible, L-extendible, and dense.

For any times $u, v, w \in T$:
Transitivity:
if $u < v$ and $v < w$ then $u < w$
Anti-symmetry:
not ($u < v$ and $v < u$)
R-totality
if $u < v$ and $u < w$ then $v < w$ or $v = w$ or $w < v$
L-totality:
if $v < u$ and $w < u$ then $v < w$ or $v = w$ or $w < v$
R-extendibility
$u < t$ for some t
L-extendibility
$t < u$ for some t
density
if $u < v$ then $u < t$ and $t < v$ for some t

Here are the clauses in the definition of truth-in-a-model-at-a-time for the tense operators:

$$V_M(\mathcal{F}\phi, t) = 1 \text{ iff there is } u \in T \text{ such that } t < u \text{ and } V_M(\phi, u) = 1$$
$$V_M(\mathcal{G}\phi, t) = 1 \text{ iff for all } u \in T \text{ such that } t < u,\ V_M(\phi, u) = 1$$
$$V_M(\mathcal{P}\phi, t) = 1 \text{ iff there is } u \in T \text{ such that } u < t \text{ and } V_M(\phi, u) = 1$$
$$V_M(\mathcal{H}\phi, t) = 1 \text{ iff for all } u \in T \text{ such that } u < t,\ V_M(\phi, u) = 1$$

Truth in a model

A sentence is true in the model if it is true at the model's present moment.

Logical consequence

$\Gamma \vDash_{TL} \phi$ iff for all TL-models M, if $V_M(\Gamma) = 1$ then $V_M(\phi) = 1$.

A New Counterexample

- I said that the past/future barrier to entailment wasn't controversial, but actually that's not *quite* true.
- A.N. Prior—whose attack on the *is/ought* barrier we've already seen—writes, in a section entitled "Correction of Hume on Past and Future" in chapter 3 of *Past, Present and Future*:

> "J.F. Bennett recently described Leibniz as having discovered, and Hume as having re-discovered, the principle that 'if Q is an immediate consequence of P then there cannot be a time-reference in Q later than the latest time-reference in P'. One thing that the development of tense-logic makes quite clear—if it was not clear before—is that this alleged 'discovery' is in fact a falsehood." (Prior, 1967, 57))

$$\mathcal{P}p \vDash \mathcal{F}\mathcal{P}p$$

Future-switching

The next thing we need is an intermodel relation (by analogy with model-extension, in the FOL case.)

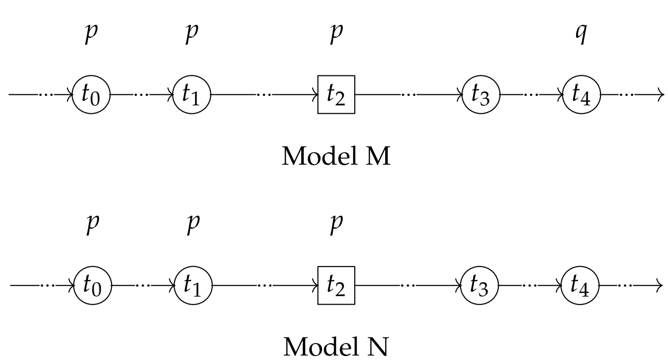

Figure 1: Two models that stand in the basic future-switch relation.

Definition 2 (Basic Future-Switching ($\curlyvee$))**.** *Let M, N be tense logic models. N is a basic future-switch of M ($M \curlyvee N$) just in case*

1. $T_N = T_M$

2. $<_M = <_N$
3. $n_M = n_N$
4. *and for all sentence letters* ϕ, *and* $t \in T_N$, *such that* $t \leq n$,

$$I_N(\phi, t) = I_M(\phi, t)$$

Past and Future Sentences

A sentence is Past iff it is Υ-preserved, Future iff it is Υ-fragile.

More general approach to fragility and preservation:

Let R be a binary relation on the set of models for a logic.

Then a sentence ϕ is R-fragile iff for all models M where $V(\phi)_M = 1$, there is some N such that MRN and $V_N(\phi) = 0$.

And a sentence is R-preserved iff for some model M where $V(\phi)_M = 1$, there is some N such that MRN and $V_N(\phi) = 0$.

Taxonomy

Past sentences: Pp, p, Hp, all logical truths
Future sentences: Fp, Gp, $Fp \wedge Pq$
Neither: $Pp \vee Fq$
Both: all unsatisfiable sentences.

Proof

— we could prove this the way we approached the proof for the Particular/Universal barrier
— that proof would be structurally the same.
— so let's do something a bit more general.

Theorem 3 (General Barrier Theorem)**.** *Suppose we have a logic, L (a formal language with a set of models) and a binary relation on those models, R. Then if* Γ *is a satisfiable set of R-preserved sentences and* ϕ *is an R-fragile sentence,* $\Gamma \not\models_L \phi$.

Proof. Let M be a model of Γ. Either ϕ is true in M or not. If not, then M is a countermodel and $\Gamma \not\models_L \phi$. So suppose ϕ is true in M. Since ϕ is R-fragile there is a model N such that MRN and ϕ is false in N. But the members of Γ are R-preserved and so Γ is true in N. So N is a countermodel and $\Gamma \not\models_L \phi$. So $\Gamma \not\models_L \phi$ either way. □

Our Past/Future barrier is the instance of this where the logic is TL and R is future-switching.

Yesterday's Particular/Universal barrier is the instance of this where the logic is FOL and R is model-extension.

What happens to Prior's other counterexample?

$\mathcal{FP}p$ turns out to be Neither (Past nor Future.)

So $Pp \models FPp$ is not a counterexample because the conclusion isn't really Future (it's not future-switch fragile.)

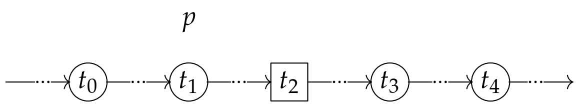

Figure 2: $\mathcal{FP}p$ is true in this model and no future-switch makes it false.

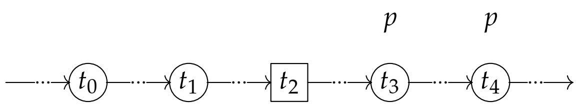

Figure 3: $\mathcal{FP}p$ is true in this model and some future-switches make it false

References

Hume, D. (1975/1748). *Enquiries concerning human understanding and concerning the principles of morals*. Oxford University Press, Oxford. edited by L. A. Selby-Bigge.

Prior, A. N. (1967). *Past, Present and Future*. Clarendon Press, Oxford.